What can quantum cosmology say about the inflationary universe?

Gianluca Calcagni
Instituto de Estructura de la Materia, CSIC, calle Serrano 121, 28006 Madrid, Spain

Claus Kiefer
Institut für Theoretische Physik, Universität zu Köln, Zülpicher Strasse 77, 50937 Köln, Germany

Christian F. Steinwachs
Physikalisches Institut, Albert-Ludwigs-Universität Freiburg, Hermann-Herder-Strasse 3, 79104 Freiburg, Germany

Abstract. We propose a method to extract predictions from quantum cosmology for inflation that can be confronted with observations. Employing the tunneling boundary condition in quantum geometrodynamics, we derive a probability distribution for the inflaton field. A sharp peak in this distribution can be interpreted as setting the initial conditions for the subsequent phase of inflation. In this way, the peak sets the energy scale at which the inflationary phase has started. This energy scale must be consistent with the energy scale found from the inflationary potential and with the scale found from a potential observation of primordial gravitational waves. Demanding a consistent history of the universe from its quantum origin to its present state, which includes decoherence, we derive a condition that allows one to constrain the parameter space of the underlying model of inflation. We demonstrate our method by applying it to two models: Higgs inflation and natural inflation.

1. Introduction

It is generally assumed that the Universe underwent a period of quasi-exponential expansion very early in its evolution. This phase is called *inflation* and has the advantage of giving a causal explanation for structure formation (see e.g. [1] for a textbook introduction). But while the kinematic features of inflation are well understood, its precise dynamical origin remains unclear. There exist plenty of different models, mostly phenomenologically devised, and one must at present resort to observations in order to constrain the class of allowed models [2].

How can one select inflationary models from a theoretical point of view? The ideal situation would be to have an established fundamental theory at one's disposal from which one can derive the dynamics of inflation, for example in the form of an inflaton field φ and its potential $V(\varphi)$. Unfortunately, we do not have a theory of this kind. This is partly related to the open question of constructing a consistent and empirically correct quantum theory of gravity [3]. While the energy scale of inflation is most likely separated from the Planck scale by some orders of magnitude, its origin can probably not be entirely understood without reference to quantum gravity.

A conservative approach to quantum gravity, which should give reliable results at least somewhat away from the Planck scale, is the direct quantization of general relativity in standard metric variables

[3]. This approach is called quantum geometrodynamics and has the important feature that it leads back to general relativity in the semiclassical limit. One might then use this framework to derive the desired constraints on inflationary models. But how is this possible?

If applied to the cosmic regime, quantum geometrodynamics leads to the framework of quantum cosmology, with the wave function of the universe as its central concept. This wave function obeys the Wheeler–DeWitt equation [3]. One could thus attempt to derive constraints on inflation from this equation. But this is not easy, because the interpretation of the wave function in quantum cosmology is far from being straightforward. In our contribution to these Proceedings, we shall review one popular method to obtaining predictions in quantum cosmology and apply it to two models of inflation.

In the following Sec. 2, we shall briefly discuss how one can envisage predictions in quantum cosmology and how one can apply them to inflation. In Sec. 3, we present the energy scales relevant for inflation. In slow-roll models, the relevant energy scale is derived from the corresponding potential. If primordial gravitational waves can be observed, one can directly extract the inflationary energy scale, which must then be consistent with the scale derived from the potential. A third scale is the one obtained from quantum cosmology, which must be consistent with the two other scales if we adopt the criterion of selection presented in Sec. 2. In Sec. 4, we apply these considerations to the models of non-minimal Higgs inflation and natural inflation. In this, we follow closely our earlier papers [4] and [5], where more details and references can be found. We end with a brief conclusion and outlook.

2. Predictions in quantum cosmology

At the most fundamental level, quantum gravity — and therefore also quantum cosmology — is timeless [3, 6]. This is a direct consequence of the fact that general relativity does not contain any absolute notion of time; after quantization, the dynamical spacetime vanishes in the same way as the classical trajectories do in quantum mechanics. In the Wheeler–DeWitt equation, there is thus no t, and the notions of probability and probability conservation (unitarity) seem to lose their usual meaning.

How, then, can one extract predictions from quantum cosmology? So far, only heuristic ideas are available. It has been suggested that a minimal scheme is to look for peaks in the wave function and to interpret them as predictions, see, for example, [7, 8]. If the wave function vanishes in a certain region of configuration space, this means that the corresponding values will never occur; this property is important for the discussion of singularity avoidance.

Inflation is a (semi)classical concept, so a prerequisite for obtaining inflation from quantum cosmology is an efficient quantum-to-classical transition. This is achieved by decoherence, a process that is well understood and experimentally explored in quantum mechanics [9, 10]. It has been shown that decoherence is efficient at the "onset of inflation", which thus justifies the use of robust semiclassical components of the universal wave function and the neglect of interference terms, see, for example, [3, 11] and the references therein. It has been suggested that one should interpret the wave function only in the semiclassical limit [12], but we leave this as an open question. For the purpose of this contribution, we adopt the heuristic proposal that a strong peak in the wave function is interpreted as a prediction, while we will not attempt to infer anything from it in the general case. In the semiclassical limit, from the wave function after decoherence one can get also a restriction on the allowed classical trajectories, that is, one obtains a selection criterion for trajectories.

In the general case, one could envisage a derivation of probabilities from quantum entanglement in the manner attempted in [13], see also [10], but this is still an open issue.

The idea to get a probability for inflation from the wave function in this way was entertained already in [14]. A more precise formulation was obtained in [15] and later papers (see the references in [4]) by emphasizing in particular the need to go to the one-loop level in quantum field theory in order to obtain a normalizable wave function. At the tree level, the slow-roll approximation does, in general, not lead to a peak because of the small field derivatives.

Even in the semiclassical limit, the form of the wave function will depend on the employed boundary condition. The two most popular boundary conditions are the 'no-boundary' and the 'tunneling'

conditions; see, for example, [3, 8] for detailed reviews. In general, the no-boundary condition will not predict the occurrence of inflation. This is why attention is concentrated on the tunneling proposal. It should be emphasized that 'tunneling' is meant here only as a metaphor because tunneling has no meaning in a timeless context, except in a limited sense in the semiclassical approximation; see [11, 16, 17] for a more detailed discussion. We shall also use the tunneling boundary condition in our contribution and shall see in which sense one can get a prediction for inflation from it. Instead of using the Wheeler–DeWitt equation, we shall employ the equivalent path-integral formulation and its semiclassical limit.

The issues of probability and probability measure are even more subtle and contrived in the recently discussed 'multiverse' context (consult, for example, [18] and the references therein), but this will not be addressed here.

3. Energy scales of inflation

In this section, we will summarize how to extract in detail the energy scale of inflation from inflationary models themselves, from observation, and from quantum cosmological considerations.

3.1. Slow-roll predictions for the energy scale of inflation

For all inflationary models, the main observables are the power spectra of primordial scalar and tensor perturbations which are generated during inflation on top of an already existing homogeneous and isotropic Friedmann-Robertson-Walker background space-time,

$$\mathcal{P}_{\rm t} := A_{\rm t}\left(\frac{k}{k_*}\right)^{n_{\rm t}}, \qquad \mathcal{P}_{\rm s} := A_{\rm s}\left(\frac{k}{k_*}\right)^{n_{\rm s}-1}. \tag{1}$$

The mode k_* corresponds to a pivot scale k_*^{-1} (to be chosen according to the observational window of the experiment) when the mode k_* first crosses the Hubble scale, $k_* = a_*\,H_*$. Here, $H(t) = \dot{a}(t)/a(t)$ denotes the Hubble parameter. Within the slow-roll approximation, deviations from a perfect de Sitter stage can be parametrized to first order in terms of the two slow-roll parameters

$$\epsilon_{\rm v} := \frac{M_{\rm P}^2}{2}\left(\frac{V'}{V}\right)^2, \qquad \eta_{\rm v} := M_{\rm P}^2\,\frac{V''}{V}\,, \tag{2}$$

where $M_{\rm P}$ denotes the (reduced) Planck mass. The amplitudes of the power spectra are given by $A_{\rm t}$ and $A_{\rm s}$. The tensor and scalar spectral indices $n_{\rm t}$ and $n_{\rm s}$ encode the scale dependence of the power spectra (its slope). These parameters can be entirely expressed in terms of V, $\epsilon_{\rm v}$, and $\eta_{\rm v}$

$$A_{\rm t} = \frac{2\,V}{3\,\pi^2\,M_{\rm P}^4}, \qquad A_{\rm s} = \frac{V}{24\,\pi^2\,M_{\rm P}^4\,\epsilon_{\rm v}}, \qquad n_t = -2\,\epsilon_{\rm v}, \qquad n_{\rm s} = 1 + 2\,\eta_{\rm v} - 6\,\epsilon_{\rm v}\,. \tag{3}$$

All quantities in (1)–(2) must be evaluated for the value of the inflaton field at Hubble-scale exit, φ_*, which, in turn, can be expressed in terms of the number of e-folds N_* by integrating and inverting the relation

$$N_* = \int_{t_*}^{t_{\rm end}} {\rm d}t\,H \simeq \frac{1}{M_P^2}\int_{\varphi_*}^{\varphi_{end}} {\rm d}\varphi\,\frac{V}{V'}\,. \tag{4}$$

The value of $\varphi_{\rm end}$, where inflation ends, is defined by the breakdown of the slow-roll approximation, when $\epsilon_{\rm v} \simeq \mathcal{O}(1)$, which leads to the convention

$$\epsilon_{\rm v}(\varphi_{\rm end}) := 1\,. \tag{5}$$

The energy scale predicted by inflationary slow-roll models is then given by

$$E_{\rm infl}^{\rm model} := V_*^{1/4} := [V(\varphi_*)]^{1/4}\,. \tag{6}$$

3.2. Observational constraints for the energy scale of inflation

The observational energy scale of inflation $E_{\rm inf}^{\rm obs}$ is unknown and so far there only exists an upper bound. Observations of primordial gravitational waves that leave their imprint in the B-polarization spectrum of the cosmic microwave radiation would allow to determine $E_{\rm inf}^{\rm obs}$ in a model independent way. But increasing precision will lead to stronger bounds and eventually even to a detection that would allow to uniquely fix $E_{\rm inf}^{\rm obs}$. In the following, we will derive how $E_{\rm infl}^{\rm obs}$ can be expressed in terms of observable quantities.

The amplitude of the scalar power spectrum $A_{\rm s}$ is fixed by the measured temperature anisotropies of the CMB, $A_{\rm s} \propto (\Delta T/T)^2$. For the pivot scale $k_* = 0.05\ {\rm Mpc}^{-1}$, the best PLANCK fit by the ΛCDM model for the scalar amplitude in the absence of tensor modes and with lensing and polarization data is [20]

$$A_{\rm s*} = (2.139 \pm 0.063) \times 10^{-9} \tag{7}$$

at the 68% confidence level.

The tensor-to-scalar ratio — to first order in the slow roll approximation — is defined as

$$r := \frac{A_{\rm t}}{A_{\rm s}} = 16\,\epsilon_{\rm v} = -8\,n_{\rm t}\,. \tag{8}$$

The B-polarization spectrum of the CMB is produced only by tensorial perturbations. A detection of B-modes would allow one to fix $A_{\rm t\,*}$ and with (7) also r_*. Finally, this would allow us to determine the energy scale of inflation in a model-independent way from observations,

$$(E_{\rm infl}^{\rm obs})^4 := \frac{3\,\pi^2\,M_{\rm P}^4}{2}\,A_{\rm t\,*} = \frac{3\,\pi^2\,M_{\rm P}^4}{2}\,A_{\rm s\,*}\,r_*. \tag{9}$$

So far, observations managed to obtain only an upper bound on r_*.

The current bound from [19] is $r_* < 0.12$ at 95% confidence level at $k_* = 0.05\,{\rm Mpc}^{-1}$. Taking the central values $A_{\rm s\,*} = 2.2 \times 10^{-9}$ and saturating the bound for r, one obtains an upper bound for the energy scale

$$E_{\rm infl}^{\rm obs} < 1.9 \times 10^{16}\,{\rm GeV}\,. \tag{10}$$

Obviously, all cosmological models have to satisfy the condition

$$E_{\rm infl}^{\rm model} \approx E_{\rm infl}^{\rm obs} \tag{11}$$

in order not to be in conflict with observational data.

3.3. Quantum cosmological energy scale of inflation

As discussed in Sec. 2, we use a heuristic approach and interpret peaks in the tunneling probability distributions as setting the initial conditions for inflation. The tunneling distribution in the semiclassical limit is found to be [4, 5]

$$\mathcal{T}(\varphi) := e^{-\Gamma(\varphi)} = \exp\left[-\frac{24\,\pi^2\,M_{\rm P}^4}{V(\varphi)}\right]\,. \tag{12}$$

A peak corresponds to a maximum of (12). Finding this peak is equivalent to finding the maxima of the potential $V_{\rm max} := V_{\rm eff}(\varphi_{\rm max})$. This leads to the simple conditions

$$\frac{{\rm d}\,V(\varphi)}{{\rm d}\varphi}\Big|_{\varphi=\varphi_{\rm max}} = 0, \qquad \frac{{\rm d}^2\,V(\varphi)}{{\rm d}\varphi^2}\Big|_{\varphi=\varphi_{\rm max}} < 0\,. \tag{13}$$

The peak $\varphi_{\max}$ in (12) corresponds to the value of φ that selects the most probable value of $\Lambda_{\text{eff}} = V(\varphi_{\max})/M_{\text{P}}^2$ for which the universe starts after tunneling. In this way, the quantum scale of inflation was obtained in [21, 22, 23, 24, 25]. The predictability of the tunneling distribution (12) can be quantified by the sharpness of the peak at $\varphi_{\max}$,

$$\mathcal{S} := \frac{(\Delta\varphi)^2}{E_{\text{infl}}^{\text{QC}}}. \tag{14}$$

Here, the variance $(\Delta\varphi)^2$ is a measure of the width of the peak, while $E_{\text{infl}}^{\text{QC}}$ is a measure for the height of the peak. We can get a rough estimate of the variance by fitting a normal distribution around the peak $\varphi_{\max}$. Taking $\varphi_{\max}$ as the mean value and expanding Γ to second order around $\varphi_{\max}$, we obtain

$$(\Delta\varphi)^2 := [\Gamma''(\varphi_{\max})]^{-1}\,. \tag{15}$$

In the inflationary slow-roll regime, $\varphi \approx \text{const}$ and the energy density is completely dominated by $V_{\max} := V(\varphi_{\max})$. Therefore, the peak value $\varphi_{\max}$ allows one to determine the energy scale of inflation as

$$E_{\text{infl}}^{\text{QC}} := V_{\max}^{1/4}\,. \tag{16}$$

Demanding a consistent quantum cosmological history of the universe, beginning with the quantum creation via tunneling, we extend the consistency condition (11) and require

$$E_{\text{infl}}^{\text{QC}} \approx E_{\text{infl}}^{\text{model}} \approx E_{\text{infl}}^{\text{obs}}\,. \tag{17}$$

This implies that the energy scale of the inflationary model must not only agree with present observations but must also be of the same order as the prediction from quantum cosmology.

Two points deserve further discussion. First, in the presence of multiple maxima of the effective potential, there might be several (possibly degenerated) peaks in the probability distribution. In such a case, the environment of these peaks, in particular the neighboring minima, has to be investigated. Second, it should be noted that in the context of eternal inflation and the landscape picture [26, 27], such a strong condition as $E_{\text{infl}}^{\text{QC}} \approx E_{\text{infl}}^{\text{obs}}$ might not hold and one has to resort to the somewhat weaker condition $E_{\text{infl}}^{\text{QC}} \geq E_{\text{infl}}^{\text{obs}}$. Even if there was only one global maximum of the effective potential, corresponding to the unique single highest peak in the tunneling distribution, one must be careful in case the effective potential also features several metastable minima with different energy densities. Then, starting from the global maximum, inflation could happen in a cascade-like process decaying from higher vacua to lower ones step by step (in the following labeled by $\varphi_{\min,n}$ where higher n mean lower values of $V(\varphi_{\min,n})$) or decaying directly in some lower metastable (or eventually even stable) vacuum. We would not be sure whether 'our inflation,' which produced our exponentially blown patch of the universe we can observe today, was due to the initial inflationary phase that started at the global maximum of the effective potential or due to another inflationary period that started in a lower metastable minimum of the effective potential. In other words, it is logically possible that, depending on the structure of the effective potential, the phase of inflation triggered by the quantum creation of the universe leads to a phase of eternal inflation with $V(\varphi_{\max})$. Then, in this eternally inflating universe at some moment in time in some region of space, the inflaton field could decay in one of the metastable vacua $\varphi_{\min,n}$, starting another phase of inflation, with a different initial condition set by the local minimum of the effective potential $V(\varphi_{\min,n}) < V(\varphi_{\max})$ and this can happen several times. We cannot really say whether the energy scale of 'our' inflation is $V(\varphi_{\max})$ or $V(\varphi_{\min,n})$. Therefore, we can realistically only demand $E_{\text{infl}}^{\text{QC}} \geq E_{\text{infl}}^{\text{obs}}$.

In this context, it might be interesting to note that according to [28] inflation can exist only eternally to the future direction, but not to the past, assuming a universe that is on average always expanding (not necessarily accelerated). This supports the assumption of an initial moment of creation, in contrast to an eternally existing inflationary universe with no beginning.

4. Special models

In the following, we will apply the general method presented in the previous sections to two particular models of inflation: non-minimal Higgs inflation and natural inflation; these two scenarios are among the class of inflationary models currently favored by observational data [29, 30]. In the following, we quote the results for the 2013 data release of PLANCK.

While natural inflation already admits a quantum cosmological analysis at the tree level, quantum corrections are essential in non-minimal Higgs inflation as they lead to the formation of a strict maximum in the potential and therefore to a sharp peak in (12).

4.1. Non-minimal Higgs inflation

In the non-minimal Higgs inflation model [31, 32, 33, 34, 35, 36, 37, 38, 39], the Standard Model Higgs boson and the cosmological inflaton are identified to be one and the same scalar field φ – the Higgs inflaton. The other essential assumption of this model is a strong non-minimal coupling $\xi \sim 10^4 - 10^5$ of the Higgs inflaton to gravity.[1] The interactions relevant for cosmology can be summarized by the graviton-Higgs sector of the model,

$$S[g,\varphi] = \int \mathrm{d}^4x\, g^{1/2} \left[U(\varphi)R - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi) \right]. \tag{18}$$

The coupling to the Ricci scalar R and the Higgs potential are given by

$$U(\varphi) = \frac{1}{2}\left(M_{\mathrm{P}}^2 + \xi\varphi^2\right), \qquad V(\varphi) = \frac{\lambda}{4}\left(\varphi^2 - \nu^2\right)^2. \tag{19}$$

Here, ξ is the non-minimal coupling constant, λ the quartic self-coupling, and $\nu \simeq 246$ GeV the symmetry breaking scale. The matter sector is given by the Standard Model interaction Lagrangian

$$\mathcal{L}_{\mathrm{int}} = -\sum_\chi \frac{1}{2}\lambda_\chi\,\chi^2\,\varphi^2 - \sum_A \frac{1}{2}g_A^2\,A_\mu^2\,\varphi^2 - \sum_\Psi y_\Psi \varphi\,\bar{\Psi}\Psi. \tag{20}$$

The sum extends over scalar fields χ, vector gauge bosons A_μ and fermions Ψ. The matter content can be restricted to the dominant contributions that come from the heavy $W^\pm$ and Z bosons and the Yukawa top quark q_{t}. Their masses follow from the relations

$$m_{W^\pm}^2 = \frac{1}{4}g\,\varphi^2, \quad m_{\mathrm{Z}}^2 = \frac{1}{4}(g^2 + g'^2)\,\varphi^2, \quad m_{\mathrm{t}}^2 = \frac{1}{2}y_{\mathrm{t}}^2\,\varphi^2, \quad m_{\mathrm{H}}^2 = \lambda\,(3\varphi^2 - \nu^2), \tag{21}$$

with the electroweak and strong gauge couplings g, g' and g_{s} as well as the Yukawa top quark coupling y_{t}. This matter content results in essential quantum contributions to the effective potential —the quantity that encodes the relevant information for the cosmological analysis. Since the energy scales of the electroweak vacuum and inflation are separated by many orders of magnitude, one also needs to take into account the dependence of the coupling constants on the energy scale. In order to evaluate the coupling constants at the high energy scale of inflation, one needs to calculate the renormalization group flow that connects the electroweak scale with the energy scale of inflation [33, 34, 35]. The renormalization group

[1] We note that, for an interacting scalar field φ, a non-mininal coupling of the form $\propto \varphi^2 R$ will be unavoidably induced already at the one-loop order. Even within an effective field theoretical approach, where higher order terms are supposed to be sufficiently suppressed, consistency of the renormalization procedure would require to include such a term already from the very beginning. Regarding the strength of ξ, we note that in view of the rather small mass of the discovered Higgs boson $M_{\mathrm{H}} \simeq 126$ GeV, the condition of a large non-minimal coupling can be relaxed somewhat; see the discussion at the end of this section and, e.g., [39, 40].

running of the couplings, in turn, is encoded in the beta functions which give rise to a system of coupled ordinary differential equations that has to be solved numerically,

$$\frac{dg_i(t)}{dt} = \beta_{g_i}, \qquad g_i = (\lambda, \xi, g, g', g_{\rm s}, y_{\rm t}), \qquad \frac{dZ(t)}{dt} = \gamma Z. \tag{22}$$

Here, $t = \ln\varphi/\mu$ is the logarithmic running scale and μ is an arbitrary renormalization point. The wave function renormalization $Z(t)$ is determined by the anomalous dimension γ of the Higgs field.

In order to make use of the standard slow-roll formalism for the cosmological analysis, it is convenient to transform to the Einstein frame by a conformal transformation of the metric field and a redefinition of the Higgs inflaton,

$$\hat{g}_{\mu\nu} = \frac{2U(\varphi)}{M_{\rm P}^2} g_{\mu\nu}, \quad \left(\frac{d\hat{\varphi}}{d\varphi}\right)^2 = \frac{M_{\rm P}^2}{2}\frac{\left(U + 3U'^2\right)}{U^2}, \quad \hat{V} = \frac{M_{\rm P}^2}{4}\frac{V}{U^2}\Big|_{\varphi=\hat{\varphi}}. \tag{23}$$

The one-loop renormalization group improved effective potential in the Einstein frame reads [35]

$$\hat{V} \simeq \frac{M_{\rm P}^4}{4}\frac{\lambda}{\xi^2}\left[1 - \frac{2M_{\rm P}^2}{\xi\varphi^2} + \frac{\boldsymbol{A_I}}{16\pi^2}\ln\left(\frac{\varphi}{\mu}\right)\right], \tag{24}$$

where $\boldsymbol{A_I}$ represents the inflationary anomalous scaling [35]

$$\boldsymbol{A_I}(t) := \frac{3}{8\lambda(t)}\left[2g^4(t) + \left(g^2(t) + g'^2(t)\right)^2 - 16y_{\rm t}^4(t)\right] - 6\lambda(t). \tag{25}$$

4.1.1. Slow-roll predictions Using the slow-roll formulas of Sec. 3.1 for the Einstein-frame renormalization-group-improved effective potential, and taking the derivatives with respect to the Einstein frame scalar field $\hat{\varphi}$, one can express the the slow-roll parameters in terms of the original Jordan frame variables [35]

$$\hat{\varepsilon} = \frac{M_P^2}{2}\left(\frac{1}{\hat{V}}\frac{d\hat{V}}{d\hat{\varphi}}\right)^2 = \frac{4}{3}\left(\frac{M_P^2}{\xi\,\varphi^2} + \frac{\boldsymbol{A_I}}{64\pi^2}\right)^2, \tag{26}$$

$$\hat{\eta} = \frac{M_P^2}{\hat{V}}\frac{d^2\hat{V}}{d\hat{\varphi}^2} = -\frac{4M_P^2}{3\xi\varphi^2}. \tag{27}$$

For the expressions of the remaining cosmological parameters, it is convenient to introduce the abbreviation

$$x := \frac{N\boldsymbol{A_I}}{48\pi^2}, \tag{28}$$

which can be interpreted as a measure of the strength of quantum corrections, resulting from $\boldsymbol{A_I}$. In this way, for the scalar amplitude one finds [35]

$$\hat{A}_{\rm s} = \frac{\lambda}{96\pi^2\xi^2\hat{\varepsilon}} = \frac{N^2}{72\pi^2}\frac{\lambda}{\xi^2}\left(\frac{e^x - 1}{x\,e^x}\right)^2. \tag{29}$$

The scalar spectral index and the tensor-to-scalar ratio are then found to be [35]

$$n_s = 1 - \frac{2}{N}\frac{x}{e^x - 1}, \tag{30}$$

$$r = \frac{12}{N^2}\left(\frac{xe^x}{e^x - 1}\right)^2. \tag{31}$$

All these quantities have to be evaluated at the energy scale of of inflation, the moment of first horizon crossing, when the inflaton value is φ_*. This, in turn, means that one has to numerically integrate the system of renormalization-group equations (22) from the electroweak vacuum $t_{\rm ew} \simeq 0$ up to the scale t_*, corresponding to φ_*, and then evaluate all running couplings at t_*.

Fixing the arbitrary renormalization point at the top mass scale $\mu = M_{\rm t}$ and assuming a modified convention for the condition of the end of inflation $\hat{\varepsilon}|_{t=t_{\rm end}} := 3/4$ (instead of the convention (5)), the times t_* and $t_{\rm end}$ can be determined via the relation $\varphi_{*/{\rm end}} = M_t \exp(t_{*/{\rm end}})$ and the estimate for the number of e-folds [32],

$$N_* \simeq \frac{3}{4}\frac{\xi(t_*)}{M_{\rm P}^2}\left(\varphi_*^2 - \varphi_{\rm end}^2\right). \tag{32}$$

In the large ξ approximation, these times read [35]

$$t_* = \ln\frac{M_{\rm P}}{M_{\rm t}} + \frac{1}{2}\ln\frac{4N}{3\xi_*} + \frac{1}{2}\ln\frac{\exp x_* - 1}{x_*}, \tag{33}$$

$$t_{\rm end} = \ln\frac{M_{\rm P}}{M_{\rm t}} + \frac{1}{2}\ln\frac{4}{3\xi_{\rm end}}. \tag{34}$$

While we have taken into account a running ξ, numerically the running is very slow, i.e. $\xi(t_{\rm ew}) \simeq \xi(t_*)$. Nevertheless, in view of the fact that there is no initial condition for $\xi(t_{\rm ew})$, one has to impose a 'final condition' $\xi(t_*)$, determined by the correct normalization of the scalar amplitude (29) evaluated at t_*, i.e. $A_{{\rm s}*} \propto \lambda_*/\xi_*^2 \propto (\Delta T/T)^2 \sim 10^{-10}$. Note that for $N = 50 \div 60$, the duration of inflation in terms of the logarithmic scale t is numerically very short $t_* - t_{\rm end} \simeq 2$ compared to the post-inflationary running $t_{\rm end} - t_{\rm ew} \simeq 35$ [35].

The numerical predictions for (30) and (31) depend on the details of the renormalization-group flow and are, in particular, very sensitive to the initial conditions at the electroweak scale. A more precise analysis including beta functions up to two and three loops has become available, see e.g. [36, 37] and, since the discovery of the Higgs boson, also the initial conditions at the electroweak scale (in particular the top mass $M_{\rm t}$) are known to a higher precision. Another aspect is connected to the rather light value of the measured Higgs mass $M_{\rm H} \simeq 126\,{\rm GeV}$. As a consequence, $\lambda(t_{\rm in})$ can be very small by itself at the energy scale of inflation and therefore allows for much smaller non-minimal couplings $\xi(t_{\rm in})$; see e.g. a discussion in [39, 40]. For certain initial values the renormalization group flow can even drive λ to negative values and therefore lead to an unstable (or meta-stable) vacuum, see e.g. [41]. In general, the precise inflationary predictions of this model are very sensible to small changes in initial values for the Standard Model masses.

4.1.2. Quantum cosmological predictions Following the general method presented in Sec. 3.3, the tunneling amplitude for the non-minimal Higgs inflation model is given by [4]

$$\Gamma(\varphi) = 24\pi^2\frac{M_{\rm P}^4}{\hat{V}(\varphi)} \simeq 96\pi^2\frac{\xi^2}{\lambda}\left(1 + \frac{2M_{\rm P}^2}{\xi Z^2\varphi^2}\right). \tag{35}$$

The peak position $\varphi_{\rm max}$ is determined by the extrema

$$\varphi\frac{d\Gamma}{d\varphi} = \frac{d\Gamma}{dt} = -\frac{6\xi^2}{\lambda}\left(\boldsymbol{A_I} + \frac{64\pi^2 M_{\rm P}^2}{\xi Z^2\varphi^2}\right) = 0, \tag{36}$$

The solution of this condition in terms of the probability peak reads [4]

$$\varphi_{\rm max}^2 = -\frac{64\pi^2 M_{\rm P}^2}{\xi\boldsymbol{A_I} Z^2}\bigg|_{t=t_{\rm max}}. \tag{37}$$

The peak is very narrow, as can be estimated by the sharpness

$$\mathcal{S} = \frac{(\Delta\varphi)^2}{E_{\rm inf}^{\rm QC}} \simeq \left.\frac{\frac{d^2\Gamma(t)}{dt^2}}{\hat{V}(t)}\right|_{t=t_{\rm max}} = -\left.\frac{\lambda}{12\xi^2}\frac{1}{\boldsymbol{A_I}}\right|_{t=t_{\rm max}} \sim 10^{-10}. \tag{38}$$

In view of $\boldsymbol{A_I}(t_{\rm max}) \simeq \boldsymbol{A_I}(t_{\rm end})$, the point of the horizon crossing φ_* for the pivot scale k, chosen to correspond to $N = 60$, is very close to the point of quantum creation $\varphi_{\rm max}$. Their ratio for different modes, corresponding to different N, therefore takes the form [4]

$$\frac{\varphi_*^2}{\varphi_{\rm max}^2} = 1 - \exp\left[-N\frac{|\boldsymbol{A_I}(t_{\rm end})|}{48\pi^2}\right]. \tag{39}$$

Thus, (39) indicates that, for wavelengths longer than the pivotal one, the instant of horizon crossing approaches the moment of ‘creation’ of the Universe, but it is always posterior to it ($\varphi_{\rm max} > \varphi_*$), as required for a consistent quantum cosmological history of the universe.

4.2. Natural inflation

Another inflationary model which is in agreement with the Planck data is that of natural inflation [42]. The inflaton potential for natural inflation reads

$$V = \Lambda^4\,\left[1 + \cos\left(\varphi/f\right)\right]\,. \tag{40}$$

In this model, φ is supposed to be a pseudo Nambu–Goldstone boson taking values on a circle with radius f and angle $\varphi/f \in [0,\,2\,\pi)$ [42]. The two constants Λ and f determine the height and the slope of the potential and have physical dimension of mass in natural units. The interpretation of φ as a pseudo Nambu–Goldstone field suggests that $f = O(M_{\rm P})$ and $\Lambda \approx M_{\rm GUT} \sim 10^{16}$ GeV.

4.2.1. Slow-roll predictions The cosmological parameters in the inflationary slow-roll analysis can again be derived from the general expressions in Sec. 3.1. From (2), the first two slow-roll parameters read

$$\epsilon_{\rm v} = \frac{M_{\rm P}^2}{2\,f^2}\,\tan^2\left[\varphi/(2\,f)\right], \qquad \eta_{\rm v} = -\frac{M_{\rm P}^2\,\cos(\varphi/f)}{f^2\,\left[1+\cos(\varphi/f)\right]}. \tag{41}$$

The scalar spectral index and the tensor-to-scalar ratio then take the form

$$n_{\rm s} = -\frac{M_{\rm P}^2}{f^2}\,\frac{3-\cos(\varphi/f)}{1+\cos(\varphi/f)}, \qquad r = \frac{8\,M_{\rm P}^2}{f^2}\,\tan^2\left[\varphi/(2\,f)\right]\,. \tag{42}$$

All cosmological observables must again be evaluated at φ_*, the field value that corresponds to the moment where the pivot mode k_* first crosses the Hubble scale. The number of e-folds N_* connecting the end of inflation $\varphi_{\rm end}$ with the value φ_* is

$$N_* = \frac{2\,f^2}{M_{\rm P}^2}\,\ln\left[\frac{\sin\left(\frac{\varphi_{\rm end}}{2\,f}\right)}{\sin\left(\frac{\varphi_*}{2\,f}\right)}\right]\,. \tag{43}$$

The value $\varphi_{\rm end}$ that determines the upper integration bound in (43) is determined to be

$$\varphi_{\rm end} = 2\,f\,\arctan(\sqrt{2}\,f/M_{\rm P})\,. \tag{44}$$

Inserting (44) in (43), solving for φ_* and parametrizing f in units of $M_{\rm P}$, we find

$$\varphi_* = 2M_{\rm P}\,\alpha\,\arcsin\left(\frac{\alpha\, e^{-N_*/2\alpha^2}}{\sqrt{1/2+\alpha^2}}\right), \tag{45}$$

where $\alpha := f/M_{\rm P}$. Evaluating the potential (40) at φ_* yields

$$V(\varphi_*) = 2\,\Lambda^4\,\left[1-\delta_V(\alpha,N_*)\right]\,, \tag{46}$$

where, following [5], we have defined

$$\delta_V(N_*,\alpha) := \frac{2\,e^{-N_*/\alpha^2}\,\alpha^2}{1+2\,\alpha^2}\,. \tag{47}$$

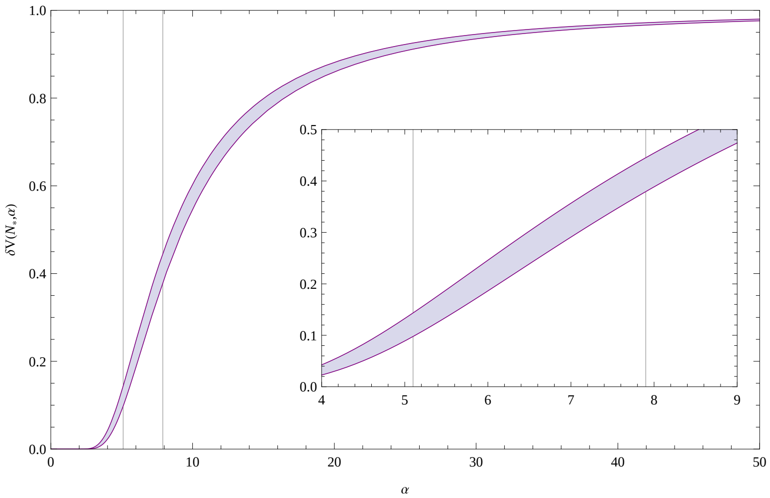

Figure 1. The function $\delta_V(N_*,\alpha)$ as a function of α for values of $N_*\in[50,60]$, taken from [5]. The upper line corresponds to $N_*=50$, the lower line to $N_*=60$. The inset shows the region with α in the 68% CL range $5.1<\alpha<7.9$ (see (49)).

The expressions for $n_{\rm s}$ and r evaluated at φ_* can be expressed in terms of δ_V and α:

$$n_{{\rm s}\,*} = 1+\frac{1}{\alpha^2}\,\frac{\delta_V(N_*,\,\alpha)+1}{\delta_V(N_*,\alpha)-1}\,,\qquad r_* = \frac{8}{\alpha^2}\,\frac{\delta_V(N_*,\,\alpha)}{1-\delta_V(N_*,\,\alpha)}\,. \tag{48}$$

For $N_*=60$, PLANCK 2013 data [29] constrain α to lie in the interval [43]

$$5.1<\alpha<7.9 \quad (68\%\,{\rm CL})\,. \tag{49}$$

4.2.2. Quantum cosmological predictions Following again the general algorithm of Sec. 3.3, we first have to calculate the extrema of (40),

$$\left.\frac{{\rm dV}}{{\rm d}\varphi}\right|_{\varphi=\varphi_{\rm ext}} = -\frac{\Lambda^4}{f}\,\sin\left(\varphi_{\rm ext}/f\right)=0\,. \tag{50}$$

If $\varphi_{\rm ext}$ is a maximum, peak values correspond to

$$\varphi_{\rm max} := 2\,\pi n f\,. \tag{51}$$

The potential at $\varphi_{\rm max}$ has the value

$$V_{\rm max} = 2\,\Lambda^4\,. \tag{52}$$

With the width $\Delta\varphi$ of the distribution given by

$$(\Delta\varphi)^2 = \frac{1}{\Gamma''}\bigg|_{\varphi=\varphi_{\rm max}} = \frac{1}{6\,\pi^2}\,\frac{f^2\,\Lambda^4}{M_{\rm P}^4}\,, \tag{53}$$

the sharpness of the peak $\varphi_{\rm max}$ is estimated as

$$\mathcal{S} = \frac{(\Delta\varphi)^2}{V_{\rm max}^{1/2}} \approx \frac{1}{6\,\pi^2}\,\frac{f^2\,\Lambda^2}{M_{\rm P}^4} \sim \frac{\Lambda^2}{M_{\rm P}^2} \sim 10^{-4}\,, \tag{54}$$

where we have used $f \sim M_{\rm P}$ and $E_{\rm inf}^{\rm QC} \sim \Lambda$, according to (16) and (52).

As can be inferred from Fig. 2, it was shown in [5] that the requirement of a deviation from the approximate consistency requirement (17) of not more than one order of magnitude leads to the following constraint for the parameter α:

$$\alpha \lesssim 710 \;\; \text{for} \;\; N_* = 50 \qquad \text{and} \qquad \alpha \lesssim 780 \;\; \text{for} \;\; N_* = 60. \tag{55}$$

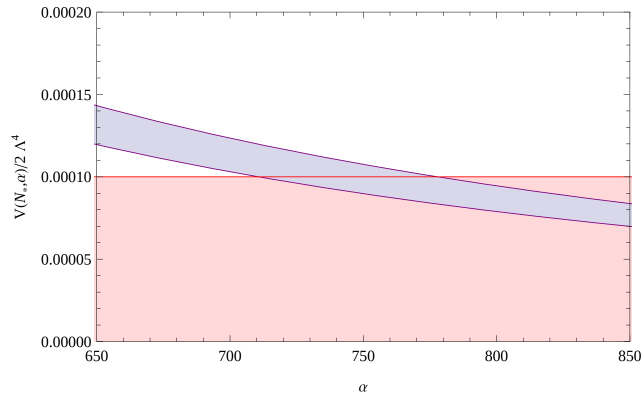

Figure 2. A zoomed-in region of the function $V(N_*,\alpha)/(2\,\Lambda^4) = 1 - \delta_V$ as a function of α for values of $N_* \in [50, 60]$, taken from [5]. The upper purple line corresponds to $N_* = 60$, the lower purple line to $N_* = 50$. The lower area, colored in light red (in black-and-white printing: light gray), corresponds to the region where $E_{\rm inf}^{\rm model} < 10^{-1}\,E_{\rm inf}^{\rm QC}$.

Although a quantum cosmological bound on α derived in this way depends on the allowed tolerance for a violation of (17), this bound is clearly not as restrictive as the constraints on α coming from the comparison with the observational constraints of the spectral index and the tensor-to-scalar ratio by PLANCK. As shown in Fig. 1, in this range $\delta_V \approx 0.1 \div 0.5$ is still small enough to respect the approximate condition (17) to the tolerated accuracy.

Thus, consistency of classical inflationary predictions with observational data (49) result in a much sharper bound $\alpha \sim O(10)$ far below the threshold $\alpha \approx 700$. We can therefore conclude that to a good approximation no conflict with the a quantum origin or our universe does arise in natural inflation, since the consistency condition is satisfied for all experimentally allowed values of α.

5. Conclusions and outlook

In this contribution, we have presented a general method that allows one to derive predictions from quantum cosmology by assuming a consistent history of our universe from its initial quantum creation up to its present state. We have restricted our analysis here to the tunneling scenario, but the method can also be extended to other quantum initial conditions such as the no-boundary condition, although this condition does not lead naturally to inflationary initial conditions.

We have in detail investigated two particular models of inflation: non-minimal Higgs inflation [31, 32, 33, 34, 35, 36, 37, 38, 39] and natural inflation [42]. We have found that both models allow for a consistent cosmic history starting from a quantum tunneling process.

In principle, all inflationary single-field models favored by recent PLANCK data can be summarized by the general class of scalar-tensor theories with the action

$$S = \int \mathrm{d}^4x \, \sqrt{g} \, \left[U(\varphi)\, R - \frac{G(\varphi)}{2} \, (\nabla\varphi)^2 - V(\varphi) \right] , \tag{56}$$

and the method presented here could, in principle, be applied also to this general class parametrized in terms of the arbitrary functions $U(\varphi)$, $G(\varphi)$ and $V(\varphi)$. As has been discussed in the context of non-minimal Higgs inflation, quantum corrections can become important and modify the shape and the location of the extrema of the effective potential. The one-loop divergences for the general action (56), necessary for renormalization, were obtained in [44] in a closed form for an even more general setup of a symmetric $O(N)$ invariant multiple of scalar fields.

Finally, a note regarding the parametrization dependence of these quantum corrections is in order. While in the transition from the Jordan to the Einstein frame parametrizations leads to equivalent formulations at tree level, in the usual quantum field theoretical formalism such a field transformation will in general induce an off-shell parametrization dependence of the effective action [45, 46, 47]. In [45], a geometric approach to the effective action was suggested to overcome the problem with non-covariance (with respect to the configuration space of field) of the ordinary formalism. Recently, this idea has been adopted in [48] in the context of non-minimal Higgs inflation.

[1] Mukhanov V 2012 *Physical Foundations of Cosmology* third edition (Cambridge: Cambridge University Press)

[2] Martin J 2015 The observational status of cosmic inflation after Planck *Preprint* astro-ph.CO/1502.05733

[3] Kiefer C 2012 *Quantum Gravity* third edition (Oxford: Oxford University Press)

[4] Barvinsky A O, Kamenshchik A Yu, Kiefer C and Steinwachs C F 2010 Tunneling cosmological state revisited: Origin of inflation with a non-minimally coupled Standard Model Higgs inflaton *Phys. Rev.* D **81** 043530 (*Preprint* hep-th/0911.1408)

[5] Calcagni G, Kiefer C and Steinwachs C F 2014 Quantum cosmological consistency condition for inflation *J. Cosmol. Astropart. Phys.* **10** 026

[6] Kiefer C 2013 Conceptual problems in quantum gravity and quantum cosmology *ISRN Math. Phys.* **2013** 509316 (*Preprint* gr-qc/1401.3578)

[7] Hartle J B 1987 *Prediction in quantum cosmology* (Gravitation in astrophysics) ed B Carter and J B Hartle (New York: Plenum Press) pp 329–60

[8] Halliwell J J 1991 *Introductory lectures on quantum cosmology* (Quantum cosmology and baby universes) ed S Coleman, J B Hartle, T Piran and S Weinberg (Singapore: World Scientific) pp 159–243

[9] Joos E, Zeh H D, Kiefer C, Giulini D, Kupsch J and Stamatescu I O 2003 *Decoherence and the Appearance of a Classical World in Quantum Theory* second edition (Berlin: Springer)

[10] Schlosshauer M 2007 *Decoherence and the quantum-to-classical transition* (Berlin: Springer)

[11] Zeh H D 2007 *The Physical Basis of the Direction of Time* fifth edition (Berlin: Springer)

[12] Vilenkin A 1989 Interpretation of the wave function of the universe *Phys. Rev.* D **39** 1116

[13] Zurek W H 2005 Probabilities from entanglement, Born's rule $p_k = |\psi_k|^2$ from envariance *Physical Review* A **71** 052105

[14] Hawking S W and Page D N 1986 Operator ordering and the flatness of the Universe *Nucl. Phys.* B **264** 185

[15] Barvinsky A O and Kamenshchik A Yu 1990 One loop quantum cosmology: the normalizability of the Hartle–Hawking wave function and the probability of inflation *Class. Quantum Grav.* **7** L181

[16] Conradi H D 1998 Tunneling of macroscopic universes *Int. J. Mod. Phys.* D **7** 189

[17] Mithani A T and Vilenkin A 2015 Tunneling decay rate in quantum cosmology *Preprint* hep-th/1503.00400

[18] Page D N 2014 Spacetime average density (SAD) cosmological measures *J. Cosmol. Astropart. Phys.* **11** 038

[19] Ade P A R *et al.* 2015 A joint analysis of BICEP2/Keck Array and Planck data *Preprint* astro-ph.CO/1502.00612

[20] Ade P A R *et al.* 2015 Planck 2015 results. XIII. Cosmological parameters *Preprint* astro-ph.CO/1502.01589

[21] Barvinsky A O and Kamenshchik A Yu 1996 Quantum origin of the early universe and the energy scale of inflation *Int. J. Mod. Phys.* D **5** 825-844 (*Preprint* gr-qc/9510032)

[22] Barvinsky A O, Kamenshchik A Yu and Mishakov I V 1997 Quantum origin of the early inflationary universe *Nucl. Phys.* B **491** 387-426 (*Preprint* gr-qc/9612004)

[23] Barvinsky A O 1999 Open inflation from quantum cosmology with a strong nonminimal coupling *Nucl. Phys.* B **561** 159-187 (*Preprint* gr-qc/9812058)

[24] Barvinsky A O and Kamenshchik A Yu 1998 Effective equations of motion and initial conditions for inflation in quantum cosmology *Nucl. Phys.* B **532** 339-360 (*Preprint* hep-th/9803052)

[25] Barvinsky A O, Kamenshchik A Yu and Kiefer C 1990 Origin of the inflationary universe *Mod. Phys. Lett.* A **14** 1083 (*Preprint* gr-qc/9905098)

[26] Carr B (ed) 2007 *Universe or Multiverse?* (Cambridge: Cambridge University Press)

[27] Douglas M R 2003 The statistics of string/M theory vacua *J. High Energy Phys.* **05** 046

[28] Borde A, Guth A H and Vilenkin A 2003 Inflationary space-times are incomplete in past directions *Phys. Rev. Lett.* **90** 151301 (*Preprint* gr-qc/0110012)

[29] Ade P A R *et al.* Planck Collaboration 2013 Planck 2013 results. XXII. Constraints on inflation *Preprint* astro-ph.CO/1303.5082

[30] Ade P A R *et al.* Planck Collaboration 2015 Planck 2015. XX. Constraints on inflation *Preprint* astro-ph.CO/1502.02114

[31] Bezrukov F L and Shaposhnikov M 2008 The Standard Model Higgs boson as the inflaton *Phys. Lett. B* **659** 703-706 (*Preprint* hep-th/0710.3755)

[32] Barvinsky A O, Kamenshchik A Yu and Starobinsky A A 2008 Inflation scenario via the Standard Model Higgs boson and LHC *J. Cosmol. Astropart. Phys* **11** 021 (*Preprint* hep-ph/0809.2104)

[33] Bezrukov F L, Magnin A and Shaposhnikov M 2009 Standard Model Higgs boson mass from inflation *Phys. Lett. B* **675** 88-92 (*Preprint* hep-ph/0812.4950)

[34] De Simone A, Hertzberg M P and Wilczek F 2009 Running inflation in the Standard Model *Phys. Lett. B* **678** 1-8 (*Preprint* hep-ph/0812.4946)

[35] Barvinsky A O, Kamenshchik A Yu, Kiefer C, Starobinsky A A and Steinwachs C 2009 Asymptotic freedom in inflationary cosmology with a non-minimally coupled Higgs field *J. Cosmol. Astropart. Phys* **12** 003 (*Preprint* hep-ph/0904.1698)

[36] Bezrukov F and Shaposhnikov M 2009 Standard Model Higgs boson mass from inflation: two loop analysis *J. High Energy Phys.* **07** 089 (*Preprint* hep-ph/0904.1537)

[37] Bezrukov F, Magnin A, Shaposhnikov M and Sibiryakov S 2011 Higgs inflation: consistency and generalisations *J. High Energy Phys.* **01** 016 (*Preprint* hep-ph/1008.5157)

[38] Barvinsky A O, Kamenshchik A Yu, Kiefer C, Starobinsky A A and Steinwachs C F 2012 Higgs boson, renormalization group, and naturalness in cosmology *Eur. Phys. J. C* **72** 2219 (*Preprint* hep-ph/0910.1041)

[39] Allison K 2014 Higgs xi-inflation for the 125-126 GeV Higgs: a two-loop analysis *J. High Energy Phys.* **1402** 040 (*Preprint* hep-ph/1306.6931)

[40] Bezrukov F, Rubio J and Shaposhnikov M 2014 Living beyond the edge: Higgs inflation and vacuum metastability *Preprint* hep-ph/1412.3811

[41] Buttazzo D, Degrassi G, Giardino P P, Giudice G F, Sala F, Salvio A and Strumia A 2013 Investigating the near-criticality of the Higgs boson *J. High Energy Phys.* **1312** 089 (*Preprint* hep-ph/1307.3536)

[42] Freese L, Frieman J A, Olinto A V 1990 Natural inflation with pseudo-Nambu–Goldstone bosons *Phys. Rev. Lett.* **65** 3233-3236

[43] Tsujikawa S, Ohashi J, Kuroyanagi S and De Felice A 2013 Planck constraints on single-field inflation *Phys. Rev.* D **88** 023529 (*Preprint* astro-ph.CO/1305.3044)

[44] Steinwachs C F and Kamenshchik A Yu 2011 One-loop divergences for gravity non-minimally coupled to a multiplet of scalar fields: calculation in the Jordan frame. I. The main results *Phys. Rev.* D **84** 024026 (*Preprint* gr-qc/1101.5047)

[45] Vilkovisky G A 1984 The unique effective action in quantum field theory *Nucl.Phys.* B **234** 125-137

[46] Steinwachs C F and Kamenshchik A Yu 2012 Non-minimal Higgs inflation and frame dependence in cosmology *AIP Conf.Proc.* **1514** 161-164 (*Preprint* gr-qc/1301.5543)

[47] Kamenshchik A Yu and Steinwachs C F 2014 Frame dependence of quantum corrections in cosmology (*Preprint* gr-qc/1408.5769)

[48] Moss I G 2014 Covariant one-loop quantum gravity and Higgs inflation *Preprint* hep-th/1409.2108